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NOTE

On the Degree of L_p Approximation with Positive Linear Operators

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The degree of approximation in L_p-spaces by positive linear operators is estimated in terms of the integral modulus of smoothness. It is shown that the conjectured optimal degree of approximation is not attained in the class of functions having a second derivative belonging to L_p. © 1996 Academic Press, Inc.

Let $\{L_n\}$ be a uniformly bounded sequence of positive linear operators from $L_p[a, b]$ into $L_p[c, d]$, $1 \leqslant p < \infty$, $a \leqslant c < d \leqslant b$. Let $\lambda_{np} = \max_{i=0,1,2} \|L_n(t^i, x) - x^i\|_p$ and assume $\lambda_{np} \to 0$ as $n \to \infty$. The conjectured optimal estimate for the rate of convergence to $f \in L_p[a, b]$ by $\{L_n(f)\}$ is

$$\|f - L_n(f)\|_p \leqslant C_p(\|f\|_p \lambda_{np} + w_{2,p}(f, \lambda_{np}^{1/2})), \tag{1}$$

where the L_p norm on the left is taken over $[c, d]$, $c_p > 0$ is independent of f and n, and $w_{2,p}$ denotes the second-order modulus of smoothness of f measured in $L_p[a, b]$. The estimate, (1), implies [4, p. 293],

$$\|f - L_n(f)\|_p \leqslant C_p(\|f\|_p \lambda_{np} + w_{r,p}(f, \lambda_{np}^{1/r})), \tag{2}$$

where $r \geqslant 3$ is an integer and $w_{r,p}$ is the rth order modulus of smoothness of f measured in $L_p[a, b]$.

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Berens and DeVore [1] have shown that (1) is valid for positive linear contraction operators from $L_1[a, b]$ to $L_1[a, b]$. The purpose of this paper is to show that (2) is always valid while (1), in general, is not.

Define the sequence $\{L_n\}$ from $L_p[0, 1]$ to $L_p[0, 1]$ by

$$L_n(f(t), x) = \begin{cases} f(x) & \text{if } \left|x - \frac{1}{2}\right| > \frac{1}{n} \\ \frac{n}{2} \int_{-1/n}^{1/n} f(x+u)\, du & \text{if } \left|x - \frac{1}{2}\right| \leqslant \frac{1}{n}. \end{cases}$$

An easy computation shows that $\lambda_{np} = \frac{1}{3}(1/n)^{2+1/p}$. Choose $f(x) = (x - \frac{1}{2})_+$. Then it is easy to verify that $\|L_n((t-\frac{1}{2})_+, x) - (x-\frac{1}{2})_+\|_p$ is asymptotically equivalent $(n \to \infty)$ to $(1/n)^{1+1/p}$.

If (1) were valid then, since $w_{2,p}((x-\frac{1}{2})_+, \delta) = O(\delta^{1+1/p})(\delta \to 0^+)$,

$$\begin{aligned} \|L_n((t-\tfrac{1}{2})_+, x) - (x-\tfrac{1}{2})_+\|_p &= O(\lambda_{np}^{(1/2)(1+1/p)}) && (n \to \infty) \\ &= O(n^{-(1+3/2p+1/2p^2)}) && (n \to \infty) \\ &= O(n^{-1(1+1/p)}) && (n \to \infty), \end{aligned}$$

which is a contradiction.

In [2], Berens and DeVore consider quantitative estimates for the degree of L_p approximation by positive linear operators in a multidimensional setting. A consequence of Theorem 3 of [2] for the one-dimensional case is, for any $f \in L_p[a, b]$,

$$\|f - L_n(f)\|_p \leqslant C_p(\|f\|_p\, \lambda_{np}^{2p/(2p+1)} + w_{2,p}(f, \lambda_{np}^{p/(2p+1)})). \tag{3}$$

The example given above can also be used to show that (3) is sharp. In [5] the authors show that (3) can be improved for certain classes of operators.

Let $L_p^{(r)}[a, b]$ denote the linear space of functions which together with their first $r-1$ derivatives, are absolutely continuous on $[a, b]$ and are such that the rth derivative is in $L_p[a, b]$. We have

THEOREM. *Let $\{L_n\}$ be a uniformly bounded sequence of positive linear operators from $L_p[a, b]$ into $L_p[c, d]$, $1 \leqslant p < \infty$, $a \leqslant c < d \leqslant b$. If $r \geqslant 3$ is an integer, then, for $f \in L_p[a, b]$,*

$$\|f - L_n(f)\|_p \leqslant C_p(\|f\|_p\, \lambda_{np} + w_{r,p}(f, \lambda_{np}^{1/r})),$$

where the L_p norm on the left is taken over $[c, d]$, $C_p > 0$ is independent of f and n, and $w_{r,p}$ is the r th order modulus of smoothness of f measured in $L_p[a, b]$.

Proof. Let $f \in L_p^{(r)}[a, b]$. Then for $t \in [a, b]$ and $x \in [c, d]$,

$$f(t) - f(x) = f'(x)(t-x) + \int_x^t (t-u)\, f''(u)\, du.$$

Thus,

$$|L_n((f(t)-f(x)), x)| \leqslant \|f'\|_\infty \cdot |L_n((t-x), x)| + \|f''\|_\infty \cdot L_n((t-x)^2, x). \tag{4}$$

By [3, Theorem 3.1], there is a constant, $k_p > 0$, such that, for $j = 0, 1, ..., r-1$,

$$\|f^{(j)}\|_\infty \leqslant k_p(\|f\|_p + \|f^{(r)}\|_p). \tag{5}$$

Consequently, by (4) and (5), for $f \in L_p^{(r)}[a, b]$,

$$\begin{aligned} \|f - L_n(f)\|_p &\leqslant \|f\|_\infty \cdot \|L_n(1, x) - 1\|_p + \|L_n((f(t)-f(x), x)\|_p \\ &\leqslant \|f\|_\infty \cdot \|L_n(1, x) - 1\|_p + \|f'\|_\infty \cdot \|L_n((t-x), x)\|_p \\ &\quad + \|f''\|_\infty \cdot \|L_n((t-x)^2, x)\|_p \\ &\leqslant k_p(\|f\|_p + \|f^{(r)}\|_p)\, \lambda_{np}. \end{aligned}$$

An application of Peetre's *K*-functional [4, p. 300] completes the proof of the theorem.

Remarks. The above theorem, for $p = 1$, is a special case of Theorem 2 of [2]. The example used to show that (3) is sharp appears in [2]. It is used there for a different purpose.

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